Categorical semantics of the reduction of GATs to two-sorted GATs.

Notes on my 4.5-month internship at the LIX

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What is a GAT ?

- A **list** of declarations

➜ Enables to declare sorts, objects of those sorts and equalities between those objects

- A **syntactic object**

➜ Type judgements are defined with induction rules

- Describes **models**

➜ Defines a category of models

A GAT for a function in Set

$$\left.\begin{array}{l} A : \mathcal{S}\text{et} \\ B : \mathcal{S}\text{et} \end{array}\right\} \text{sorts}$$

- -

$$\text{exec} : A \to B \quad \}\text{constructors}$$

Models: triples (A,B,f)

A GAT for an bijective function in Set

$$A : \mathcal{S}\text{et}$$

$$B : \mathcal{S}\text{et}$$

} sorts

$$\text{exec} : A \to B$$

$$\text{invexec} : B \to A$$

} constructors

$$\text{isol} : (x : A) \to \text{invexec}(\text{exec}\ \ x) = x$$

$$\text{isor} : (y : B) \to \text{exec}(\text{invexec}\ \ y) = y$$

} equalities

Models: triples (A,B,f) s.t. f is bijective

A GAT for a small category

$\mathcal{O}\text{bj} : \mathcal{S}\text{et}$

$\mathcal{H}\text{om} : \mathcal{O}\text{bj} \to \mathcal{O}\text{bj} \to \mathcal{S}\text{et}$

$\text{id} : (A : \mathcal{O}\text{bj}) \to \mathcal{H}\text{om}\ A\ A$

$\circ : (A\ B\ C : \mathcal{O}\text{bj}) \to \mathcal{H}\text{om}\ B\ C \to \mathcal{H}\text{om}\ A\ B \to \mathcal{H}\text{om}\ A\ C$

$\text{idl} : (AB : \mathcal{O}\text{bj}) \to (\sigma : \mathcal{H}\text{om}\ A\ B) \to \circ\ (\text{id}\ \ B)\ \sigma = \sigma$

$\text{idr} : (AB : \mathcal{O}\text{bj}) \to (\sigma : \mathcal{H}\text{om}\ A\ B) \to \circ\ \sigma\ (\text{id}\ \ A) = \sigma$

$\circ\text{-trans} : \ldots$

Models:
small categories

A GAT for Type Theory

$\mathsf{Con} : \mathcal{S}\mathsf{et}$

$\mathsf{Ty} : \mathsf{Con} \to \mathcal{S}\mathsf{et}$

$\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\ \Gamma \to \mathcal{S}\mathsf{et}$

$\mathsf{empty} : \mathsf{Con}$

$\mathsf{ext} : (\Gamma : \mathsf{Con}) \to (A : \mathsf{Ty}\ \Gamma) \to \mathsf{Con}$

$\mathsf{implies} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\ \Gamma \to \mathsf{Ty}\ \Gamma \to \mathsf{Ty}\ \Gamma$

$\mathsf{app} : (\Gamma : \mathsf{Con}) \to (A\ B : \mathsf{Ty}\ \Gamma) \to$
$\quad \mathsf{Tm}\ \Gamma\ (\mathsf{implies}\ A\ B) \to \mathsf{Tm}\ \Gamma\ A \to \mathsf{Tm}\ \Gamma\ B$

Models : Triples $(X_{\mathsf{Con}}, X_{\mathsf{Ty}}, X_{\mathsf{Tm}})$ with constructors $\mathsf{empty} \in X_{\mathsf{Con}}$

$\vdots$

Subject of my internship
2-sortification of GATs

Transform a GAT into a GAT with only two sorts

- ➥ Process observed since at least 2021
- ➥ Never formally studied
- ➥ Studying all GATs by only studying GATs with two sorts ?

2-sortification of the Set Function GAT

2-sortification of the Set Function GAT

$\mathcal{O} : \mathcal{S}\text{et}$	sorts	$o : \mathcal{O}$	$\leftrightarrow$	o is a sort
$\mathcal{E}\text{l} : \mathcal{O} \to \mathcal{S}\text{et}$	objects of that sort	$x : \mathcal{E}\text{l}\ o$	$\leftrightarrow$	$x : o$

2-sortification of the Set Function GAT

$\mathcal{O} : \mathcal{S}\mathrm{et}$	sorts	$o : \mathcal{O}$	$\leftrightarrow$	o is a sort
$\mathcal{E}\mathrm{l} : \mathcal{O} \to \mathcal{S}\mathrm{et}$	objects of that sort	$x : \mathcal{E}\mathrm{l}\ o$	$\leftrightarrow$	$x : o$

$A : \mathcal{O}$	$A : \mathcal{S}\mathrm{et}$
$B : \mathcal{O}$	$B : \mathcal{S}\mathrm{et}$
$\mathrm{exec} : \mathcal{E}\mathrm{l}\ A \to \mathcal{E}\mathrm{l}\ B$	$\mathrm{exec} : A \to B$

2-sortification of Type Theory GAT

$\mathcal{O} : \mathcal{S}\mathsf{et}$

$\mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}$

2-sortification of Type Theory GAT

$$\mathcal{O} : \mathcal{S}\mathsf{et}$$

$$\mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}$$

$$\mathsf{Con} : \mathcal{O}$$

$$\mathsf{Ty} : \mathcal{E}\mathsf{l}\ \mathsf{Con} \to \mathcal{O}$$

$$\mathsf{Tm} : (\Gamma : \mathcal{E}\mathsf{l}\ \mathsf{Con}) \to \mathcal{E}\mathsf{l}\ (\mathsf{Ty}\ \Gamma) \to \mathcal{O}$$

$$\mathsf{empty} : \mathcal{E}\mathsf{l}\ \mathsf{Con}$$

$$\mathsf{ext} : (\Gamma : \mathcal{E}\mathsf{l}\ \mathsf{Con}) \to (A : \mathcal{E}\mathsf{l}\ (\mathsf{Ty}\ \Gamma)) \to \mathcal{E}\mathsf{l}\ \mathsf{Con}$$

$$\mathsf{implies} : (\Gamma : \mathcal{E}\mathsf{l}\ \mathsf{Con}) \to \mathcal{E}\mathsf{l}\ (\mathsf{Ty}\ \Gamma) \to \mathcal{E}\mathsf{l}\ (\mathsf{Ty}\ \Gamma) \to \mathcal{E}\mathsf{l}\ (\mathsf{Ty}\ \Gamma)$$

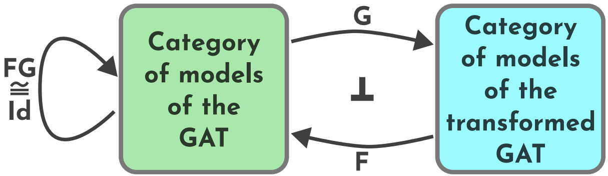
Goal of the internship

Is this transformation correct ?

➥ Can one study all GATs by studying only GATs with two sorts

How to state this fact ?

➥ Semantical proof



➥ This adjunction proves that one can make the initial model of any GAT from the initial model of the transformed GAT

Categories of models

$$\text{GAT} = \text{sorts} + \text{constructors} + \text{equalities}$$

Categories of models

$$\text{GAT} = \textbf{sorts} + \sout{\text{constructors}} + \sout{\text{equalities}}$$

$\mathsf{Con} : \mathcal{S}\mathsf{et}$ $\mathsf{Ty} : \mathsf{Con} \to \mathcal{S}\mathsf{et}$ $\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathcal{S}\mathsf{et}$	$\mathcal{O} : \mathcal{S}\mathsf{et}$ $\mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}$ - - - - - - - - - - $\underline{\mathsf{Con}} : \mathcal{O}$ $\underline{\mathsf{Ty}} : \mathcal{E}\mathsf{l}\,\underline{\mathsf{Con}} \to \mathcal{O}$ $\underline{\mathsf{Tm}} : (\Gamma : \mathcal{E}\mathsf{l}\,\underline{\mathsf{Con}}) \to \mathcal{E}\mathsf{l}(\underline{\mathsf{Ty}}\,\Gamma) \to \mathcal{O}$
$\mathcal{C} \hookrightarrow (\mathsf{Con}, \mathsf{Ty}, \mathsf{Tm})$	$\mathcal{B} \hookrightarrow (\mathcal{O}, \mathcal{E}\mathsf{l}, \underline{\mathsf{Con}}, \underline{\mathsf{Ty}}, \underline{\mathsf{Tm}})$

Categories of models

$\mathsf{Con} : \mathcal{S}\mathsf{et}$ $\mathsf{Ty} : \mathsf{Con} \to \mathcal{S}\mathsf{et}$ $\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\ \Gamma \to \mathcal{S}\mathsf{et}$	$\mathcal{O} : \mathcal{S}\mathsf{et}$ $\mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}$ - - - - - - - - - - - - - - - - $\underline{\mathsf{Con}} : \mathcal{O}$ $\underline{\mathsf{Ty}} : \mathcal{E}\mathsf{l}\ \underline{\mathsf{Con}} \to \mathcal{O}$ $\underline{\mathsf{Tm}} : (\Gamma : \mathcal{E}\mathsf{l}\ \underline{\mathsf{Con}}) \to \mathcal{E}\mathsf{l}(\underline{\mathsf{Ty}}\ \Gamma) \to \mathcal{O}$
$\mathcal{C}_0 \hookrightarrow ()$ $\mathcal{C}_1 \hookrightarrow (\mathsf{Con})$ $\mathcal{C}_2 \hookrightarrow (\mathsf{Con}, \mathsf{Ty})$ $\mathcal{C}_3 \hookrightarrow (\mathsf{Con}, \mathsf{Ty}, \mathsf{Tm})$	$\mathcal{B}_0 \hookrightarrow (\mathcal{O}, \mathcal{E}\mathsf{l})$ $\mathcal{B}_1 \hookrightarrow (\mathcal{O}, \mathcal{E}\mathsf{l}, \underline{\mathsf{Con}})$ $\mathcal{B}_2 \hookrightarrow (\mathcal{O}, \mathcal{E}\mathsf{l}, \underline{\mathsf{Con}}, \underline{\mathsf{Ty}})$ $\mathcal{B}_3 \hookrightarrow (\mathcal{O}, \mathcal{E}\mathsf{l}, \underline{\mathsf{Con}}, \underline{\mathsf{Ty}}, \underline{\mathsf{Tm}})$

Categories of Models
of the original GAT

$()$

$\mathsf{Con} : \mathcal{S}\mathsf{et}$

$\mathsf{Ty} : (\Gamma : \mathsf{Con}) \to \mathcal{S}\mathsf{et}$

$\mathsf{Tm} : (\Delta : \mathsf{Con}) \to (A : \mathsf{Ty}\ \Delta) \to \mathcal{S}\mathsf{et}$

$$
\begin{aligned}
\mathcal{C}_0 &:= \mathbf{1} \\
\mathcal{C}_1 &:= (X_{\mathsf{Con}}) \\
\mathcal{C}_2 &:= \left(X_{\mathsf{Con}}, (X_{\mathsf{Ty}}(\Gamma))_{\Gamma \in X_{\mathsf{Con}}}\right) \\
\mathcal{C}_3 &:= \Big(X_{\mathsf{Con}}, (X_{\mathsf{Ty}}(\Gamma))_{\Gamma \in X_{\mathsf{Con}}}, \\
&\qquad \left((X_{\mathsf{Tm}}(\Delta, A))_{A \in \mathsf{Ty}(\Delta)}\right)_{\Delta \in X_{\mathsf{Con}}}\Big)
\end{aligned}
$$

Categories of Models

of the original GAT

$$\boxed{()}$$
$$\boxed{\mathsf{Con} : \mathcal{S}\mathsf{et}}$$
$$\boxed{\mathsf{Ty} : (\Gamma : \mathsf{Con}) \to \mathcal{S}\mathsf{et}}$$
$$\boxed{\mathsf{Tm} : (\Delta : \mathsf{Con}) \to (A : \mathsf{Ty}\ \Delta) \to \mathcal{S}\mathsf{et}}$$

$$\begin{aligned}
\mathcal{C}_0 &:= \mathbf{1} \\
\mathcal{C}_1 &:= (X_{\mathsf{Con}} : \mathcal{S}\mathsf{et}) \\
\mathcal{C}_2 &:= \left(X_{\mathsf{Con}} : \mathcal{S}\mathsf{et}, X_{\mathsf{Ty}} : \mathcal{S}\mathsf{et}^{X_{\mathsf{Con}}}\right) \\
\mathcal{C}_3 &:= \Big(X_{\mathsf{Con}} : \mathcal{S}\mathsf{et}, X_{\mathsf{Ty}} : \mathcal{S}\mathsf{et}^{X_{\mathsf{Con}}}, \\
&\qquad X_{\mathsf{Tm}} : \mathcal{S}\mathsf{et}^{\prod_{\Delta : X_{\mathsf{Con}}} X_{\mathsf{Ty}}(\Delta)}\Big)
\end{aligned}$$

Categories of Models
of the original GAT

$$\boxed{()}$$
$$\boxed{\mathsf{Con} : \mathcal{S}\mathsf{et}}$$
$$\boxed{\mathsf{Ty} : (\Gamma : \mathsf{Con}) \to \mathcal{S}\mathsf{et}}$$
$$\boxed{\mathsf{Tm} : (\Delta : \mathsf{Con}) \to (A : \mathsf{Ty}\ \Delta) \to \mathcal{S}\mathsf{et}}$$

$$\begin{aligned}
\mathcal{C}_0 &:= \mathbf{1} \\
\mathcal{C}_1 &:= (\bullet : \mathcal{C}_0) \times (\mathcal{S}\mathsf{et}) \\
\mathcal{C}_2 &:= (X_{\mathsf{Con}} : \mathcal{C}_1) \times \left(\mathcal{S}\mathsf{et}^{X_{\mathsf{Con}}}\right) \\
\mathcal{C}_3 &:= ((X_{\mathsf{Con}}, X_{\mathsf{Ty}}) : \mathcal{C}_2) \times \\
&\qquad \left(\mathcal{S}\mathsf{et}^{\prod_{\Delta : X_{\mathsf{Con}}} X_{\mathsf{Ty}}(\Delta)}\right)
\end{aligned}$$

Categories of Models

of the original GAT

$$()$$

$$\mathrm{Con} : \mathcal{S}\mathrm{et}$$

$$\mathrm{Ty} : (\Gamma : \mathrm{Con}) \to \mathcal{S}\mathrm{et}$$

$$\mathrm{Tm} : (\Delta : \mathrm{Con}) \to (A : \mathrm{Ty}\ \Delta) \to \mathcal{S}\mathrm{et}$$

$$\begin{aligned}
\mathcal{C}_0 &:= \mathbf{1} \\
\mathcal{C}_1 &:= (X : \mathcal{C}_0) \times \mathcal{S}\mathrm{et}^{H_1(X)} \\
\mathcal{C}_2 &:= (X : \mathcal{C}_1) \times \mathcal{S}\mathrm{et}^{H_2(X)} \\
\mathcal{C}_3 &:= (X : \mathcal{C}_2) \times \mathcal{S}\mathrm{et}^{H_3(X)}
\end{aligned}$$

$$\begin{aligned}
H_1(\bullet) &= 1_{\mathcal{S}\mathrm{et}} \\
H_2(X_{\mathrm{Con}}) &= X_{\mathrm{Con}} \\
H_3(X_{\mathrm{Con}}, X_{\mathrm{Ty}}) &= \prod_{\Delta : X_{\mathrm{Con}}} X_{\mathrm{Ty}}(\Delta)
\end{aligned}$$

Category of Models
of the transformed GAT

$$\boxed{\begin{array}{c} \mathcal{O} : \mathcal{S}\mathsf{et} \\ \mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et} \end{array}}$$

$$\mathcal{B}_0 := \left(X_{\mathcal{U}} : \mathcal{S}\mathsf{et}, X_{\mathcal{E}\mathsf{l}} : \mathcal{S}\mathsf{et}^{X_{\mathcal{U}}} \right)$$

Category of Models
of the transformed GAT

$$\boxed{\begin{array}{c}\mathcal{O} : \mathcal{S}\mathsf{et} \\ \mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}\end{array}}$$

$$\mathcal{B}_0 := \left(X_{\mathcal{U}} : \mathcal{S}\mathsf{et}, X_{\mathcal{E}\mathsf{l}} : \mathcal{S}\mathsf{et}^{X_{\mathcal{U}}}\right)$$

$X_{\mathcal{U}}$:	Sorts
$X_{\mathcal{E}\mathsf{l}}(o)$:	Objects of sort o

Category of Models
of the transformed GAT

$X_{\mathcal{U}}$: Sorts
$X_{\mathcal{E}l}(o)$: Objects of sort o

$$\boxed{\mathcal{O} : \mathcal{S}\mathrm{et} \quad \mathcal{E}\mathrm{l} : \mathcal{O} \to \mathcal{S}\mathrm{et}} \qquad \mathcal{B}_0 = (X_{\mathcal{U}}, X_{\mathcal{E}l})$$

$$\boxed{\mathrm{Con} : \mathcal{O}} \qquad \mathbf{Con}_X : X_{\mathcal{U}}$$

$$\boxed{\mathrm{Ty} : (\Gamma : \underline{\mathrm{Con}}) \to \mathcal{O}} \qquad \mathbf{Ty}_X : (\Gamma \in X_{\mathcal{E}l}(\mathbf{Con}_X)) \to X_{\mathcal{U}}$$

$$\boxed{\mathrm{Tm} : (\Delta : \underline{\mathrm{Con}}) \to (A : \underline{\mathrm{Ty}\ \Delta}) \to \mathcal{O}} \qquad \mathbf{Tm}_X : \begin{array}{c}(\Delta \in X_{\mathcal{E}l}(\mathbf{Con}_X)) \to \\ (A \in X_{\mathcal{E}l}(\mathbf{Ty}_X(\Delta))) \to \\ X_{\mathcal{U}}\end{array}$$

Category of Models

of the transformed GAT

$X_{\mathcal{U}}$: Sorts

$X_{\mathcal{E}\mathsf{l}}(o)$: Objects of sort o

$$\mathcal{O} : \mathcal{S}\mathsf{et} \qquad \mathcal{E}\mathsf{l} : \mathcal{O} \to \mathcal{S}\mathsf{et}$$

$$\mathsf{Con} : \mathcal{O}$$

$$\mathsf{Ty} : (\Gamma : \underline{\mathsf{Con}}) \to \mathcal{O}$$

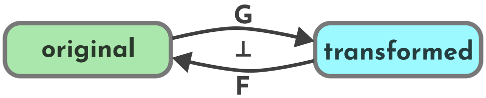
$$\mathsf{Tm} : (\Delta : \underline{\mathsf{Con}}) \to (A : \underline{\mathsf{Ty}\ \Delta}) \to \mathcal{O}$$

$$\mathcal{B}_0 = (X_{\mathcal{U}}, X_{\mathcal{E}\mathsf{l}})$$

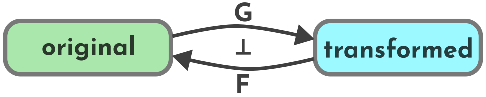
$$\mathbf{Con}_X : H_1 F_0(X) \to X_{\mathcal{U}}$$

$$\mathbf{Ty}_X : H_2 F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$$

$$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$$

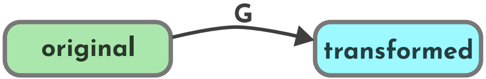


Constructing F and G



$Y : \mathcal{C}_3$	$X : \mathcal{B}_3$
	$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathsf{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$
$Y_{\mathrm{Con}} : \mathcal{S}\mathrm{et}$	$\mathbf{Con}_X : X_{\mathcal{U}}$
$(Y_{\mathrm{Ty}}(\Gamma))_{\Gamma \in Y_{\mathrm{Con}}}$	$\mathbf{Ty}_X : H_2 F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$
$\left((Y_{\mathrm{Tm}}(\Delta, A))_{A \in Y_{\mathrm{Ty}}(\Delta)}\right)_{\Delta \in Y_{\mathrm{Con}}}$	$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$

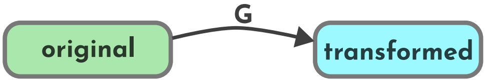
Constructing G_3



$$Y = (Y_{\mathsf{Con}}, Y_{\mathsf{Ty}}, Y_{\mathsf{Tm}})$$

$$\begin{aligned} X_{\mathcal{U}} &= \text{«sorts»} \\ X_{\mathcal{E}\mathsf{l}}(o) &= \text{«objects of sort } o\text{»} \end{aligned}$$

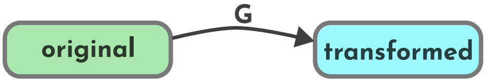
Constructing G_3



$$Y = (Y_{\mathsf{Con}}, Y_{\mathsf{Ty}}, Y_{\mathsf{Tm}})$$

$$X_{\mathcal{U}} = \{\star\} \uplus Y_{\mathsf{Con}} \uplus \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta) \xrightarrow{X_{\mathcal{E}\mathsf{l}}} \mathcal{S}\mathsf{et}$$

Constructing G_3



$$Y = (Y_{\mathsf{Con}}, Y_{\mathsf{Ty}}, Y_{\mathsf{Tm}})$$

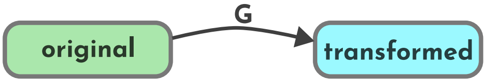
$$X_{\mathcal{U}} = \{\star\} \uplus Y_{\mathsf{Con}} \uplus \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta) \xrightarrow{X_{\mathcal{E}l}} \mathcal{S}\mathsf{et}$$

$$\star \longmapsto Y_{\mathsf{Con}}$$

$$X_{\mathcal{E}l} \qquad \Gamma \longmapsto Y_{\mathsf{Ty}}(\Gamma)$$

$$(\Delta, A) \longmapsto Y_{\mathsf{Tm}}(\Delta, A)$$

Constructing G_3



$$Y = (Y_{\mathsf{Con}}, Y_{\mathsf{Ty}}, Y_{\mathsf{Tm}})$$

$$X_{\mathcal{U}} = \{\star\} \uplus Y_{\mathsf{Con}} \uplus \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta) \xrightarrow{X_{\mathcal{E}l}} \mathcal{S}\mathsf{et}$$

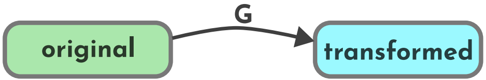
$$\star \longmapsto Y_{\mathsf{Con}}$$

$$X_{\mathcal{E}l} \qquad \Gamma \longmapsto Y_{\mathsf{Ty}}(\Gamma)$$

$$(\Delta, A) \longmapsto Y_{\mathsf{Tm}}(\Delta, A)$$

$$\begin{aligned}
\mathbf{Con}_X &= \star \in \{\star\} \\
\mathbf{Ty}_X(\Gamma) &= \Gamma \in Y_{\mathsf{Con}} \\
\mathbf{Tm}_X(\Delta, A) &= (\Delta, A) \in \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta)
\end{aligned}$$

Constructing G_3



$$Y = (Y_{\mathsf{Con}}, Y_{\mathsf{Ty}}, Y_{\mathsf{Tm}})$$

$$X_{\mathcal{U}} = \{\star\} \uplus Y_{\mathsf{Con}} \uplus \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta) \xrightarrow{X_{\mathcal{E}\mathsf{l}}} \mathcal{S}\mathsf{et}$$

$X_{\mathcal{E}\mathsf{l}}$:

$$\star \longmapsto Y_{\mathsf{Con}}$$

$$\Gamma \longmapsto Y_{\mathsf{Ty}}(\Gamma)$$

$$(\Delta, A) \longmapsto Y_{\mathsf{Tm}}(\Delta, A)$$

$$\begin{aligned} \mathbf{Con}_X &= \star \in \{\star\} \\ \mathbf{Ty}_X(\Gamma) &= \Gamma \in Y_{\mathsf{Con}} \\ \mathbf{Tm}_X(\Delta, A) &= (\Delta, A) \in \coprod_{\Delta \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Delta) \end{aligned}$$

All sorts of $X_{\mathcal{U}}$ are in the image of some constructor $(\mathbf{Con}_X, \mathbf{Ty}_X, \mathbf{Tm}_X)$

Constructing F_3

$X : \mathcal{B}_{\mathbf{3}}$	$Y = F_3(X) : \mathcal{C}_{\mathbf{3}}$
$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathrm{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$	
$\mathbf{Con}_X : X_{\mathcal{U}}$	$Y_{\mathrm{Con}} =$
$\mathbf{Ty}_X : H_2F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Ty}}(\Gamma) =$
$\mathbf{Tm}_X : H_3F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Tm}}(\Delta, A) =$

Constructing F_3

$X : \mathcal{B}_3$	$Y = F_3(X) : \mathcal{C}_3$
$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathrm{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$	
$\mathbf{Con}_X : X_{\mathcal{U}}$	$Y_{\mathrm{Con}} = X_{\mathcal{E}\mathrm{l}}(\mathbf{Con}_X)$
$\mathbf{Ty}_X : H_2 F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Ty}}(\Gamma) =$
$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Tm}}(\Delta, A) =$

Constructing F_3

$X : \mathcal{B}_3$	$Y = F_3(X) : \mathcal{C}_3$
$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathrm{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$	
$\mathbf{Con}_X : X_{\mathcal{U}}$	$Y_{\mathrm{Con}} = X_{\mathcal{E}\mathrm{l}}(\mathbf{Con}_X)$
$\mathbf{Ty}_X : H_2 F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Ty}}(\Gamma) = X_{\mathcal{E}\mathrm{l}}(\mathbf{Ty}_X(\Gamma))$
$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Tm}}(\Delta, A) =$

Constructing F_3

$X : \mathcal{B}_3$	$Y = F_3(X) : \mathcal{C}_3$
$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathrm{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$	
$\mathbf{Con}_X : X_{\mathcal{U}}$	$Y_{\mathrm{Con}} = X_{\mathcal{E}\mathrm{l}}(\mathbf{Con}_X)$
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$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Tm}}(\Delta, A) = X_{\mathcal{E}\mathrm{l}}(\mathbf{Tm}_X(\Delta, A))$

Constructing F_3

$X : \mathcal{B}_3$	$Y = F_3(X) : \mathcal{C}_3$
$X_{\mathcal{U}} : \mathcal{S}\mathrm{et} \quad X_{\mathcal{E}\mathrm{l}} : \mathcal{S}\mathrm{et}^{X_{\mathcal{U}}}$	
$\mathbf{Con}_X : X_{\mathcal{U}}$	$Y_{\mathrm{Con}} = X_{\mathcal{E}\mathrm{l}}(\mathbf{Con}_X)$
$\mathbf{Ty}_X : H_2 F_1(X, \mathbf{Con}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Ty}}(\Gamma) = X_{\mathcal{E}\mathrm{l}}(\mathbf{Ty}_X(\Gamma))$
$\mathbf{Tm}_X : H_3 F_2(X, \mathbf{Con}_X, \mathbf{Ty}_X) \to X_{\mathcal{U}}$	$Y_{\mathrm{Tm}}(\Delta, A) = X_{\mathcal{E}\mathrm{l}}(\mathbf{Tm}_X(\Delta, A))$

The preimage by $X_{\mathcal{E}\mathrm{l}}$ of each object of Y is a sort of $X_{\mathcal{U}}$ constructed by some constructor

Adjunction $F \vdash G$

$$\mathcal{H}\text{om}_{\mathcal{B}_3}\left(G_3 Y, X\right) \simeq \mathcal{H}\text{om}_{\mathcal{C}_3}\left(Y, F_3 X\right)$$

Adjunction $F \vdash G$

$$\mathcal{H}\mathrm{om}_{\mathcal{B}_3}(G_3 Y, X) \simeq \mathcal{H}\mathrm{om}_{\mathcal{C}_3}(Y, F_3 X)$$

Remark

Morphisms of $\mathcal{B}_3$ and $\mathcal{C}_3$ both respect constructors

Adjunction $F \vdash G$

$$\mathcal{H}\mathrm{om}_{\mathcal{B}_3}(G_3 Y, X) \simeq \mathcal{H}\mathrm{om}_{\mathcal{C}_3}(Y, F_3 X)$$

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All sorts of $G_3 Y_{\mathcal{U}}$ are in the image of some constructor ($\mathbf{Con}_{G_3 Y}, \mathbf{Ty}_{G_3 Y}, \mathbf{Tm}_{G_3 Y}$)
➜$\mathcal{H}\mathrm{om}_{\mathcal{B}_3}(G_3 Y, X)$ *morphisms are only defined on constructors*

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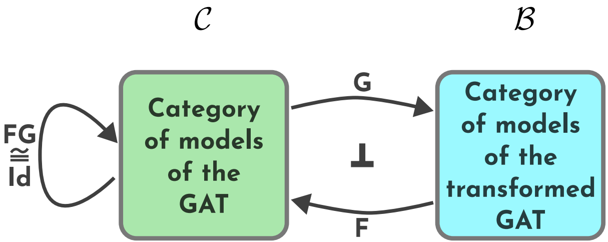
➜$\mathcal{H}\mathrm{om}_{\mathcal{B}_3}(G_3 Y, X)$ *morphisms are only defined on constructors*
The preimage by $X_{\mathcal{E}|}$ of each object of $F_3 X$ is a sort of $X_{\mathcal{U}}$ constructed by some constructor
➜$\mathcal{H}\mathrm{om}_{\mathcal{C}_3}(Y, F_3 X)$ *morphisms only send to constructible terms*

Conclusion



- Expressing GATs semantics with finite direct categories
- Construction of semantics from syntax

Future work

- Complete GAT (term constructors + equalities)
- Proof Assistant Formalization
- S_i non-direct

Thank you for your attention

Reflection

$$\begin{array}{rcl} F_3 G_3(Y)_{\mathsf{Con}} & = & G_3(Y)_p^{-1}(\{\mathcal{C}\mathsf{str}^{G_3(Y)}_{\mathsf{Con}}\}) \\ & = & G_3(Y)_p^{-1}(\{\mathsf{inj}_1 \star\}) \\ & = & Y_{\mathsf{Con}} \end{array}$$

$$\begin{array}{rcl} F_3 G_3(Y)_{\mathsf{Ty}}(\Gamma) & = & G_3(Y)_p^{-1}(\{\mathcal{C}\mathsf{str}^{G_3(Y)}_{\mathsf{Ty}}(\Gamma)\}) \\ & = & G_3(Y)_p^{-1}(\{\mathsf{inj}_2\, \Gamma\}) \\ & = & \mathsf{proj}_1^{-1}(\Gamma) \\ & = & \left\{(\Gamma', A) \in \coprod_{\Gamma' \in Y_{\mathsf{Con}}} Y_{\mathsf{Ty}}(\Gamma') \middle| \Gamma' = \Gamma\right\} \\ & \simeq & Y_{\mathsf{Ty}}(\Gamma) \end{array}$$

$$F_3 G_3(Y)_{\mathsf{Tm}}(\Delta, A) \simeq Y_{\mathsf{Tm}}(\Delta, A)$$

Structure of the global proof

- Categories $\mathcal{C}_i$ $\quad$ $\mathcal{B}_i$
- Functors $F_i : \mathcal{B}_i \to \mathcal{C}_i : G_i$
- Adjunction $F_i \vdash G_i$
- Forgetful functor $R^i_{i-1} : \mathcal{B}_i \to \mathcal{B}_{i-1}$
- Operator $\triangleleft^i : \mathcal{B}_i \times \mathcal{B}_0 \to \mathcal{B}_i$ $\quad$ $\mathrm{inj}^i_1 : X \to X \triangleleft^i Y$ $\quad$ $\mathrm{inj}^i_2 : Y \to R^i_0(X \triangleleft^i Y)$
- Coreflection $F_i G_i \cong \mathcal{I}\mathrm{d}_{\mathcal{C}_i}$
- Isomorphism $F_i\, \mathrm{inj}^i_1$
- Isomorphism $(R^i_{i-1} X) \triangleleft^{i-1} Y \to R^i_{i-1}(X \triangleleft^i Y)$

Fibration of $\mathcal{C}_i$

$\boxed{\mathsf{Con} : \mathcal{S}\mathsf{et}}$	$\boxed{\mathsf{Ty} : (\Gamma : \mathsf{Con}) \to \mathcal{S}\mathsf{et}}$	$\boxed{\mathsf{Tm} : (\Delta : \mathsf{Con}) \to (A : \mathsf{Ty}\ \Delta) \to \mathcal{S}\mathsf{et}}$
$S_1 = \quad \mathsf{Con}$	$S_2 = \quad \mathsf{Con} \leftarrow \mathsf{Ty}$	$S_3 = \quad \mathsf{Con} \leftarrow \mathsf{Ty} \leftarrow \mathsf{Tm}$
$[S_1, \mathcal{S}\mathsf{et}] \simeq \mathcal{C}_1$	$[S_2, \mathcal{S}\mathsf{et}] \simeq \mathcal{C}_2$	$[S_3, \mathcal{S}\mathsf{et}] \simeq \mathcal{C}_3$

S_i from syntax

$$T : \left[x_a^T : U_T(a)\left[x_b^{U_T(a)} \mapsto x_{v_T(a,b)}^T\right]_{b \leq I_{U_T(a)}}\right]_{a \leq I_T} \to \mathcal{S}\mathrm{et}$$

- I_T is the number of parameters of the sort T
- x_a^T is the ath parameter of the sort T
- $U_T(a)$ is the sort of the ath parameter of the sort T (we have $U_T(a) < T$)
- $I_{U_T(a)}$ is the number of parameters of the sort $U_T(a)$ i.e. the number of arguments we have to give to make an object of sort $U_T(a)$
- $x_b^{U_T(a)}$ is the bth parameter of the sort $U_T(a)$
- $v_T(a,b)$ is the index of the parameter of the sort T given as the bth parameter of the ath parameter of the sort T (we have $v_T(a,b) < a$)
- The types of parameters have to be respected, therefore we must have the equality

$$U_T(v_T(a,b)) = U_{U_T(a)}(b)$$

S_i from syntax

$$\Gamma_i(T_x) = \left\{ x_a^{T_i} \middle| a \leq I_{T_i} \text{ and } U_{T_i}(a) = T_x \right\}$$

$$\Gamma_i(x_b^{T_x} : T_x \to T_y)(x_a^{T_i} \in \Gamma_i(T_x)) = x^{T_i}_{v_{T_i}(a,b)}$$