Logic as a Second-Order Generalized Algebraic Theory

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Plan of the presentation

1. Definitions & Introduction
2. Propositional Logic
3. Conclusion

Definitions

A logic :

Formulæ Provability Proof constructors

Definitions

A logic :

Formulæ   Provability   Proof constructors

A Generalized Algebraic Theory (GAT)

- Sorts
  $\mathsf{A} \;:\; \mathbf{Set}$
- Constructors
  $\mathsf{l} \;:\; \mathsf{Int} \to \mathsf{A}$
  $\mathsf{n} \;:\; \mathsf{A} \to \mathsf{A} \to \mathsf{A}$
- Equations
  $\mathsf{eq} \;:\; \{a\ b \;:\; \mathsf{A}\} \;\to\; a \equiv b \to \mathsf{n}\ a\ b \equiv a$

Definitions

- **A Model of a Logic** : Implements every Sort, every Constructor, every Equation
- **The Minimal Model** : The Syntax

$$\text{GAT} \xrightarrow{\text{implements}} \text{Models} \xleftarrow{\text{minimal}} \text{Syntax}$$

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- **A Strict Syntax** : No quotient
- **Completeness** (of a class of models)
  Proovable in every model of this class $\leftrightarrow$ Provable in the Syntax

The real goal

Second-Order Generalized Algebraic Theory (SOGAT)

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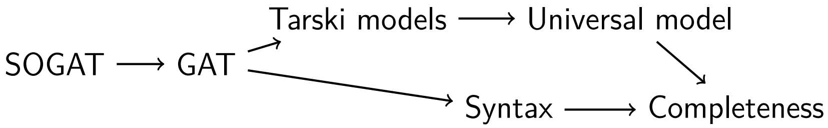
Second-Order Generalized Algebraic Theory (SOGAT)

Like GAT, but SO

- To be published
- One example to understand

Summary


SOGAT
GAT
Tarski models
Universal model
Syntax
Completeness

Informal presentation of Propositional Logic

- Formulæ are ι or implications
- Proofs are introduction or elimination of implication
- de Bruijn variables

➢ This is Simply Typed λ-calculus with de Bruijn indices and only one type

Propositional Logic as a SOGAT

$$
\begin{aligned}
\mathsf{For} &: \mathbf{Set} \\
- \Longrightarrow - &: \mathsf{For} \to \mathsf{For} \to \mathsf{For} \\
\iota &: \mathsf{For} \\
\\
\mathsf{Pf} &: \mathsf{For} \to \mathbf{Prop}^{+} \\
\mathsf{lam} &: (\mathsf{Pf}\ A \to^{+} \mathsf{Pf}\ B) \to \mathsf{Pf}\ (A \Longrightarrow B) \\
\mathsf{app} &: \mathsf{Pf}\ (A \Longrightarrow B) \to (\mathsf{Pf}\ A \to \mathsf{Pf}\ B)
\end{aligned}
$$

The base category

$$
\begin{aligned}
&\mathsf{Con} &&: \mathbf{Set}\ \ell^1\\
&\mathsf{Sub} &&: \mathsf{Con} \to \mathsf{Con} \to \mathbf{Prop}\ \ell^2\\
&\mathsf{id} &&: \{\Gamma : \mathsf{Con}\} \to \mathsf{Sub}\ \Gamma\ \Gamma\\
&\_\circ\_ &&: \mathsf{Sub}\ \Delta\ \Xi \to \mathsf{Sub}\ \Gamma\ \Delta \to \mathsf{Sub}\ \Gamma\ \Xi\\
&\diamond &&: \mathsf{Con}\\
&\varepsilon &&: \{\Gamma : \mathsf{Con}\} \to \mathsf{Sub}\ \Gamma\ \diamond
\end{aligned}
$$

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&\diamond &&: \mathsf{Con} \\
&\varepsilon &&: \{\Gamma : \mathsf{Con}\} \to \mathsf{Sub}\ \Gamma\ \diamond
\end{aligned}
$$

If $A : \mathbf{Prop}$ and $a, b : A$ then $a \equiv b$ makes no sense.

$$\mathsf{idl} : \{\Gamma\ \Delta : \mathsf{Con}\}\ \{\sigma : \mathsf{Sub}\ \Gamma\ \Delta\} \to (\mathsf{id}\ \{\Delta\}) \circ \sigma \equiv \sigma$$

$$\mathsf{Sub}\ \Gamma\ \Delta \iff \mathsf{Sub}\ \Gamma\ \Delta$$

Sorts from the SOGAT

$$\mathsf{For} : \quad \textbf{Set}$$
$$\mathsf{Pf} : \quad \mathsf{For} \to \textbf{Prop}^{+}$$

Sorts from the SOGAT

$$\begin{aligned} \mathsf{For} &: \quad \mathbf{Set} \\ \mathsf{Pf} &: \quad \mathsf{For} \to \mathbf{Prop}^{+} \end{aligned}$$

$$\begin{aligned}
&\mathsf{For} \ : \mathsf{Con} \to \mathbf{Set}\ \ell^3 \\
&_[_]\mathsf{f} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\} \to \mathsf{For}\ \Gamma \to \mathsf{Sub}\ \Delta\ \Gamma \to \mathsf{For}\ \Delta \\
&[]\mathsf{f\text{-}id} \ : \{\Gamma\ : \mathsf{Con}\} \to \{F\ : \mathsf{For}\ \Gamma\} \to F\ [\ \mathsf{id}\ \{\Gamma\}\ ]\mathsf{f} \equiv F \\
&[]\mathsf{f\text{-}\circ} : \{\Gamma\ \Delta\ \Xi\ : \mathsf{Con}\}\ \{\alpha\ : \mathsf{Sub}\ \Xi\ \Delta\} \\
&\quad \{\beta\ : \mathsf{Sub}\ \Delta\ \Gamma\}\ \{F\ : \mathsf{For}\ \Gamma\} \\
&\quad \to F\ [\ \beta \circ \alpha\ ]\mathsf{f} \equiv (F\ [\ \beta\ ]\mathsf{f})\ [\ \alpha\ ]\mathsf{f}
\end{aligned}$$

Sorts from the SOGAT

$$\mathsf{For} : \mathbf{Set}$$
$$\mathsf{Pf} : \mathsf{For} \to \mathbf{Prop}^{+}$$

$$\mathsf{For} : \mathsf{Con} \to \mathbf{Set}\ \ell^{3}$$
$$\_[\_]\mathsf{f} : \{\Gamma\ \Delta : \mathsf{Con}\} \to \mathsf{For}\ \Gamma \to \mathsf{Sub}\ \Delta\ \Gamma \to \mathsf{For}\ \Delta$$
$$[]\mathsf{f\text{-}id} : \{\Gamma : \mathsf{Con}\} \to \{F : \mathsf{For}\ \Gamma\} \to F\ [\ \mathsf{id}\ \{\Gamma\}\ ]\mathsf{f} \equiv F$$
$$[]\mathsf{f\text{-}}\circ : \{\Gamma\ \Delta\ \Xi : \mathsf{Con}\}\ \{\alpha : \mathsf{Sub}\ \Xi\ \Delta\}$$
$$\{\beta : \mathsf{Sub}\ \Delta\ \Gamma\}\ \{F : \mathsf{For}\ \Gamma\}$$
$$\to F\ [\ \beta \circ \alpha\ ]\mathsf{f} \equiv (F\ [\ \beta\ ]\mathsf{f})\ [\ \alpha\ ]\mathsf{f}$$

$$\mathsf{Pf} : (\Gamma : \mathsf{Con}) \to \mathsf{For}\ \Gamma \to \mathbf{Prop}\ \ell$$
$$\_[\_]\mathsf{p} : \{\Gamma\ \Delta : \mathsf{Con}\} \to \{F : \mathsf{For}\ \Gamma\}$$
$$\to \mathsf{Pf}\ \Gamma\ F \to (\sigma : \mathsf{Sub}\ \Delta\ \Gamma) \to \mathsf{Pf}\ \Delta\ (F\ [\ \sigma\ ]\mathsf{f})$$

Plussed Arrow from the SOGAT

$$\mathsf{Pf} : \quad \mathsf{For} \to \mathbf{Prop}^{+}$$

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$$\mathsf{Pf} : \quad \mathsf{For} \to \mathbf{Prop}^+$$

$$\_\triangleright_p\_ \; : (\Gamma \; : \mathsf{Con}) \to \mathsf{For}\ \Gamma \to \mathsf{Con}$$

$$\pi_p{}^1 \; : \{\Gamma\ \Delta \; : \mathsf{Con}\}\{F \; : \mathsf{For}\ \Gamma\}$$
$$\to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F) \to \mathsf{Sub}\ \Delta\ \Gamma$$

$$\pi_p{}^2 \; : \{\Gamma\ \Delta \; : \mathsf{Con}\}\{F \; : \mathsf{For}\ \Gamma\}$$
$$\to (\sigma \; : \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F)) \to \mathsf{Pf}\ \Delta\ (F\ [\ \pi_p{}^1\ \sigma\ ]\mathsf{f})$$

$$\_,_p\_ \; : \{\Gamma\ \Delta \; : \mathsf{Con}\}\{F \; : \mathsf{For}\ \Gamma\} \to$$
$$(\sigma \; : \mathsf{Sub}\ \Delta\ \Gamma) \to \mathsf{Pf}\ \Delta\ (F\ [\ \sigma\ ]\mathsf{f}) \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F)$$

Formulæ Constructors

$$\iota : \quad \mathsf{For}$$
$$\_\_ \Longrightarrow \_\_ : \quad \mathsf{For} \to \mathsf{For} \to \mathsf{For}$$

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$\iota \ : \{\Gamma \ : \mathsf{Con}\} \to \mathsf{For}\ \Gamma$
$[]\mathsf{f}\text{-}\iota \ : \{\Gamma\ \Delta \ : \mathsf{Con}\}\ \{\sigma \ : \mathsf{Sub}\ \Delta\ \Gamma\} \to \iota\ [\ \sigma\ ]\mathsf{f} \equiv \iota$

$\_\Rightarrow\_ \ : \{\Gamma \ : \mathsf{Con}\} \to \mathsf{For}\ \Gamma \to \mathsf{For}\ \Gamma \to \mathsf{For}\ \Gamma$
$[]\mathsf{f}\text{-}\Rightarrow \ : \{\Gamma\ \Delta \ : \mathsf{Con}\} \to \{F\ G \ : \mathsf{For}\ \Gamma\} \to \{\sigma \ : \mathsf{Sub}\ \Delta\ \Gamma\}$
$\quad \to (F \Rightarrow G)\ [\ \sigma\ ]\mathsf{f} \equiv (F\ [\ \sigma\ ]\mathsf{f}) \Rightarrow (G\ [\ \sigma\ ]\mathsf{f})$

Proof Constructors

$$\mathsf{lam} : \quad (\mathsf{Pf}\ A \to^{+} \mathsf{Pf}\ B) \to \mathsf{Pf}\ (A \Longrightarrow B)$$
$$\mathsf{app} : \quad \mathsf{Pf}\ (A \Longrightarrow B) \to (\mathsf{Pf}\ A \to \mathsf{Pf}\ B)$$

✚ Morphisms, Mappings

Proof Constructors

$$\text{lam} : \quad (\text{Pf } A \to^{+} \text{Pf } B) \to \text{Pf } (A \Longrightarrow B)$$
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```
--# Lam & App
```

$\text{lam} \;:\; \{\Gamma \;:\; \text{Con}\}\{A\; B \;:\; \text{For } \Gamma\}$
$\quad \to \text{Pf } (\Gamma \triangleright_p A)\; (B\,[\,{\pi_p}^1\ \text{id}\,]\text{f}) \to \text{Pf } \Gamma\; (A \Rightarrow B)$
$\text{app} \;:\; \{\Gamma \;:\; \text{Con}\}\{A\; B \;:\; \text{For } \Gamma\}$
$\quad \to \text{Pf } \Gamma\; (A \Rightarrow B) \to \text{Pf } \Gamma\; A \to \text{Pf } \Gamma\; B$

Proof Constructors

$$\text{lam}: \quad (\text{Pf A} \to^{+} \text{Pf B}) \to \text{Pf (A} \Longrightarrow \text{B)}$$
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```
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$\quad \to \text{Pf}\ (\Gamma \triangleright_{\text{p}} A)\ (B\ [\ {\pi_{\text{p}}}^{1}\ \text{id}\ ]\text{f}) \to \text{Pf}\ \Gamma\ (A \Rightarrow B)$
$\text{app} : \{\Gamma : \text{Con}\}\{A\ B : \text{For}\ \Gamma\}$
$\quad \to \text{Pf}\ \Gamma\ (A \Rightarrow B) \to \text{Pf}\ \Gamma\ A \to \text{Pf}\ \Gamma\ B$

✚ Morphisms, Mappings

How to make a strict syntax

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1. Make it
2. Make a morphism $I \to M$
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Idea : Make Formulæ independent from contexts

For : ~~Con~~ $\to$ **Set** ℓ^3

« Formulæ are the same whichever the context »

Defining For and Con

```
--#
data For  : Set where
  ι  : For
  _⇒_  : For → For → For

data Con  : Set where
  ◇  : Con
  _▷p_  : Con → For → Con
```

Defining Proofs

```
data PfVar  : Con → For → Prop where
  pvzero  : PfVar (Γ ▷p A) A
  pvnext  : PfVar Γ A → PfVar (Γ ▷p B) A
data Pf  : Con → For → Prop where
  var  : PfVar Γ A → Pf Γ A
  lam  : Pf (Γ ▷p A) B → Pf Γ (A ⇒ B)
  app  : Pf Γ (A ⇒ B) → Pf Γ A → Pf Γ B
```

Defining Sub

```
data Sub  : Con → Con → Prop where
  ε  : Sub Γ ◇
  _,p_  : Sub Δ Γ → Pf Δ A → Sub Δ (Γ ▷p A)
```

```
_[_]p  : Pf Γ A → Sub Δ Γ → Pf Δ A
var pvzero [ _ ,p pf ]p = pf
var (pvnext pv) [ σ ,p _ ]p = var pv [ σ ]p
lam pf [ σ ]p = lam (pf [ wkSub σ ,p var pvzero ]p)
app pf pf' [ σ ]p = app (pf [ σ ]p) (pf' [ σ ]p)
```

Defining Initial Morphism

```
--#
mCon  : Con → (ZOL.Con M)
mFor  : {Γ  : Con} → For → (ZOL.For M (mCon Γ))

mCon ◇ = ZOL.◇ M
mCon (Γ ▷p A) = ZOL._▷p_ M (mCon Γ) (mFor {Γ} A)
mFor {Γ} ι = ZOL.ι M
mFor {Γ} (A ⇒ B) = ZOL._⇒_ M (mFor {Γ} A) (mFor {Γ} B)
```

How to prove completeness

- We can define a universal model
- We need to use the initial morphism for the universal model
- Formulæ of Γ implies those of $\Delta \leftrightarrow$ I.Sub Γ Δ

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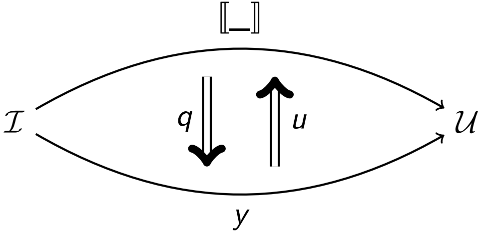

How to prove completeness

$$
\begin{aligned}
&\mathsf{completeness} : \{\Gamma\ \Delta : \mathsf{I.Con}\} \\
&\quad \to (\{\Xi : \mathsf{I.Con}\} \to [\![\ \Gamma\ ]\!]\mathsf{c}\ \Xi \to [\![\ \Delta\ ]\!]\mathsf{c}\ \Xi\ ) \\
&\quad \to \mathsf{I.Sub}\ \Gamma\ \Delta \\
&\mathsf{completeness}\ \{\Gamma\}\ \{\Delta\}\ f = \mathsf{q}\ \Delta\ \Gamma\ (f\ \{\Gamma\}\ (\mathsf{u}\ \Gamma\ \Gamma\ \mathsf{I.id}))
\end{aligned}
$$

What about predicate Logic

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A lot harder because :

1. Sub is not in **Prop** anymore, because Term are in **Set**
2. Two way of extending contexts
3. Formulæ now depend on contexts

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1. Sub is not in **Prop** anymore, because Term are in **Set**
2. Two way of extending contexts
3. Formulæ now depend on contexts

✿ We can use the same idea : Split Contexts

✷ The completeness proof is not yet done

Conclusion

What we have done

- Definition of Propositional Logic and Predicate Logic (as SOGAT)
- Tranformation from SOGAT to GAT
- Completeness proof in Category theory language

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Further work

- Initial morphism & Completeness for Predicate Logic
- Add positive operators ($\vee$, $\exists$) (need Beth models)
- Split contexts in the syntax
- One equation in the SOGAT $\equiv$ One equation in the GAT

Thank you for your patient hearing

$$
\begin{array}{lcl}
\text{For} & : & \textbf{Set} \\
\_ \Longrightarrow \_ & : & \text{For} \to \text{For} \to \text{For} \\
\text{R} & : & \text{TM} \to \text{TM} \to \text{For} \\
\forall & : & (\text{TM} \to \text{For}) \to \text{For} \\
\\
\text{Pf} & : & \text{For} \to \textbf{Prop}^{+} \\
\text{lam} & : & (\text{Pf A} \to^{+} \text{Pf B}) \to \text{Pf (A} \Longrightarrow \text{B)} \\
\text{app} & : & \text{Pf (A} \Longrightarrow \text{B)} \to (\text{Pf A} \to \text{Pf B}) \\
\forall i & : & (\text{t : TM} \to \text{Pf A t}) \to \text{Pf} (\forall \text{A}) \\
\forall e & : & \text{Pf} (\forall \text{A}) \to (\text{t : TM}) \to \text{Pf (A t)}
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{Tm} & : & \mathbf{Set}^{+} \\
\\
\mathsf{For} & : & \mathbf{Set} \\
\_ \Longrightarrow \_ & : & \mathsf{For} \to \mathsf{For} \to \mathsf{For} \\
\mathsf{R} & : & \mathsf{Tm} \to \mathsf{Tm} \to \mathsf{For} \\
\forall & : & (\mathsf{Tm} \to \mathsf{For}) \to \mathsf{For} \\
\\
\mathsf{Pf} & : & \mathsf{For} \to \mathbf{Prop}^{+} \\
\mathsf{lam} & : & (\mathsf{Pf}\ \mathsf{A} \to^{+} \mathsf{Pf}\ \mathsf{B}) \to \mathsf{Pf}\ (\mathsf{A} \Longrightarrow \mathsf{B}) \\
\mathsf{app} & : & \mathsf{Pf}\ (\mathsf{A} \Longrightarrow \mathsf{B}) \to (\mathsf{Pf}\ \mathsf{A} \to \mathsf{Pf}\ \mathsf{B}) \\
\forall i & : & (\mathsf{t} : \mathsf{Tm} \to^{+} \mathsf{Pf}\ \mathsf{A}\ \mathsf{t}) \to \mathsf{Pf}\ (\forall \mathsf{A}) \\
\forall e & : & \mathsf{Pf}\ (\forall \mathsf{A}) \to (\mathsf{t} : \mathsf{Tm}) \to \mathsf{Pf}\ (\mathsf{A}\ \mathsf{t})
\end{array}
$$

$$
\begin{aligned}
&\mathsf{Con} : \mathbf{Set}\ \ell^1 \\
&\mathsf{Sub} : \mathsf{Con} \to \mathsf{Con} \to \mathbf{Set}\ \ell \\
&\_\circ\_ : \{\Gamma\ \Delta\ \Xi : \mathsf{Con}\} \to \mathsf{Sub}\ \Delta\ \Xi \to \mathsf{Sub}\ \Gamma\ \Delta \to \mathsf{Sub}\ \Gamma\ \Xi \\
&\circ\text{-ass} : \{\Gamma\ \Delta\ \Xi\ \Psi : \mathsf{Con}\} \\
&\quad \{\alpha : \mathsf{Sub}\ \Gamma\ \Delta\}\{\beta : \mathsf{Sub}\ \Delta\ \Xi\}\{\gamma : \mathsf{Sub}\ \Xi\ \Psi\} \\
&\quad \to (\gamma \circ \beta) \circ \alpha \equiv \gamma \circ (\beta \circ \alpha) \\
&\mathsf{id} : \{\Gamma : \mathsf{Con}\} \to \mathsf{Sub}\ \Gamma\ \Gamma \\
&\mathsf{idl} : \{\Gamma\ \Delta : \mathsf{Con}\}\ \{\sigma : \mathsf{Sub}\ \Gamma\ \Delta\} \to (\mathsf{id}\ \{\Delta\}) \circ \sigma \equiv \sigma \\
&\mathsf{idr} : \{\Gamma\ \Delta : \mathsf{Con}\}\ \{\sigma : \mathsf{Sub}\ \Gamma\ \Delta\} \to \sigma \circ (\mathsf{id}\ \{\Gamma\}) \equiv \sigma \\
&\diamond : \mathsf{Con} \\
&\varepsilon : \{\Gamma : \mathsf{Con}\} \to \mathsf{Sub}\ \Gamma\ \diamond \\
&\varepsilon\text{-u} : \{\Gamma : \mathsf{Con}\} \to \{\sigma : \mathsf{Sub}\ \Gamma\ \diamond\} \to \sigma \equiv \varepsilon\ \{\Gamma\}
\end{aligned}
$$

$$
\begin{aligned}
&\_\triangleright_{\mathsf{t}} \ : \mathsf{Con} \to \mathsf{Con}\\
&\pi_{\mathsf{t}}{}^{1} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\} \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_{\mathsf{t}}) \to \mathsf{Sub}\ \Delta\ \Gamma\\
&\pi_{\mathsf{t}}{}^{2} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\} \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_{\mathsf{t}}) \to \mathsf{Tm}\ \Delta\\
&\_,_{\mathsf{t}}\_ \ : \{\Gamma\ \Delta\ : \mathsf{Con}\} \to \mathsf{Sub}\ \Delta\ \Gamma \to \mathsf{Tm}\ \Delta \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_{\mathsf{t}})\\
&\pi_{\mathsf{t}}{}^{2}\circ,_{\mathsf{t}} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\}\ \{\sigma\ : \mathsf{Sub}\ \Delta\ \Gamma\}\ \{t\ : \mathsf{Tm}\ \Delta\} \to\\
&\quad \pi_{\mathsf{t}}{}^{2}\ (\sigma\ ,_{\mathsf{t}}\ t) \equiv t\\
&\pi_{\mathsf{t}}{}^{1}\circ,_{\mathsf{t}} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\}\ \{\sigma\ : \mathsf{Sub}\ \Delta\ \Gamma\}\ \{t\ : \mathsf{Tm}\ \Delta\}\\
&\quad \to \pi_{\mathsf{t}}{}^{1}\ (\sigma\ ,_{\mathsf{t}}\ t) \equiv \sigma\\
&,_{\mathsf{t}}\circ\pi_{\mathsf{t}} \ : \{\Gamma\ \Delta\ : \mathsf{Con}\}\ \{\sigma\ : \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_{\mathsf{t}})\}\\
&\quad \to (\pi_{\mathsf{t}}{}^{1}\ \sigma)\ ,_{\mathsf{t}}\ (\pi_{\mathsf{t}}{}^{2}\ \sigma) \equiv \sigma\\
&,_{\mathsf{t}}\circ : \{\Gamma\ \Delta\ \Xi\ : \mathsf{Con}\}\{\sigma\ : \mathsf{Sub}\ \Gamma\ \Xi\}\{\delta\ : \mathsf{Sub}\ \Delta\ \Gamma\}\{t\ : \mathsf{Tm}\ \Gamma\}\\
&\quad \to (\sigma\ ,_{\mathsf{t}}\ t) \circ \delta \equiv (\sigma \circ \delta)\ ,_{\mathsf{t}}\ (t\ [\ \delta\ ]\mathsf{t})
\end{aligned}
$$

$$
\begin{aligned}
&\_\triangleright_p\_ : (\Gamma : \mathsf{Con}) \to \mathsf{For}\ \Gamma \to \mathsf{Con} \\
&\pi_p{}^1 : \{\Gamma\ \Delta : \mathsf{Con}\}\{F : \mathsf{For}\ \Gamma\} \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F) \to \mathsf{Sub}\ \Delta\ \Gamma \\
&\pi_p{}^2 : \{\Gamma\ \Delta : \mathsf{Con}\}\{F : \mathsf{For}\ \Gamma\} \to \\
&\quad (\sigma : \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F)) \to \Delta \vdash (F\ [\ \pi_p{}^1\ \sigma\ ]\mathrm{f}) \\
&\_,_p\_ : \{\Gamma\ \Delta : \mathsf{Con}\}\{F : \mathsf{For}\ \Gamma\} \to (\sigma : \mathsf{Sub}\ \Delta\ \Gamma) \to \\
&\quad \Delta \vdash (F\ [\ \sigma\ ]\mathrm{f}) \to \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F) \\
&\quad ,_p\circ\pi_p : \{\Gamma\ \Delta : \mathsf{Con}\}\{F : \mathsf{For}\ \Gamma\}\{\sigma : \mathsf{Sub}\ \Delta\ (\Gamma \triangleright_p F)\} \to \\
&\quad (\pi_p{}^1\ \sigma)\ ,_p\ (\pi_p{}^2\ \sigma) \equiv \sigma \\
&\pi_p{}^1\circ,_p : \{\Gamma\ \Delta : \mathsf{Con}\}\{\sigma : \mathsf{Sub}\ \Delta\ \Gamma\}\{F : \mathsf{For}\ \Gamma\} \\
&\quad \{prf : \Delta \vdash (F\ [\ \sigma\ ]\mathrm{f})\} \to \pi_p{}^1\ (\sigma\ ,_p\ prf) \equiv \sigma
\end{aligned}
$$

$$
\begin{aligned}
&,_p\circ : \{\Gamma\ \Delta\ \Xi : \mathsf{Con}\}\{\sigma : \mathsf{Sub}\ \Gamma\ \Xi\}\{\delta : \mathsf{Sub}\ \Delta\ \Gamma\}\{F : \mathsf{For}\ \Xi\}\{prf : \Gamma \vdash (F\ [\ \sigma\ ]\mathrm{f})\} \\
&\quad \to (\sigma\ ,_p\ prf) \circ \delta \equiv (\sigma \circ \delta)\ ,_p\ (\mathsf{substP}\ (\ F \to \Delta \vdash F)\ (\equiv\!\mathsf{sym}\ []\mathrm{f}\text{-}\circ)\ (prf\ [\ \delta\ ]\mathrm{p}))
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{Tm} : \mathsf{Con} \to \mathbf{Set}\ \ell^2 \\
&\_[\_]\mathsf{t} : \{\Gamma\ \Delta : \mathsf{Con}\} \to \mathsf{Tm}\ \Gamma \to \mathsf{Sub}\ \Delta\ \Gamma \to \mathsf{Tm}\ \Delta \\
&[]\mathsf{t\text{-}id} : \{\Gamma : \mathsf{Con}\} \to \{x : \mathsf{Tm}\ \Gamma\} \to x\ [\ \mathsf{id}\ \{\Gamma\}\ ]\mathsf{t} \equiv x \\
&[]\mathsf{t\text{-}}\circ : \{\Gamma\ \Delta\ \Xi : \mathsf{Con}\}\ \{\alpha : \mathsf{Sub}\ \Xi\ \Delta\}\ \{\beta : \mathsf{Sub}\ \Delta\ \Gamma\} \\
&\quad \{t : \mathsf{Tm}\ \Gamma\} \to t\ [\ \beta \circ \alpha\ ]\mathsf{t} \equiv (t\ [\ \beta\ ]\mathsf{t})\ [\ \alpha\ ]\mathsf{t}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{lam} : \{\Gamma : \mathsf{Con}\}\{F\ G : \mathsf{For}\ \Gamma\} \\
&\quad \to (\Gamma \rhd_{\mathsf{p}} F) \vdash (G\ [\ {\pi_{\mathsf{p}}}^1\ \mathsf{id}\ ]\mathsf{f}) \to \Gamma \vdash (F \Rightarrow G) \\
&\mathsf{app} : \{\Gamma : \mathsf{Con}\}\{F\ G : \mathsf{For}\ \Gamma\} \\
&\quad \to \Gamma \vdash (F \Rightarrow G) \to \Gamma \vdash F \to \Gamma \vdash G
\end{aligned}
$$

```
--# Term contexts are isomorphic to Nat
data Cont : Set where
  ◇t : Cont
  _▷t⁰ : Cont → Cont
```

```
data TmVar  : Cont → Set  where
  tvzero  : TmVar (Γ_t ▷t⁰)
  tvnext  : TmVar Γ_t → TmVar (Γ_t ▷t⁰)

data Tm  : Cont → Set  where
  var  : TmVar Γ_t → Tm Γ_t
```

```
data For  : Cont → Set  where
  R  : Tm Γ_t → Tm Γ_t → For Γ_t
  _⇒_  : For Γ_t → For Γ_t → For Γ_t
  ∀∀  : For (Γ_t ▷t⁰) → For Γ_t
```

```
data Subt  : Cont → Cont → Set  where
  εt  : Subt Γt ◇t
  _,t_  : Subt Δt Γt → Tm Δt → Subt Δt (Γt ▷t⁰)
```

$$\_[\_]\mathsf{t} \;:\; \mathsf{Tm}\ \Gamma_t \to \mathsf{Subt}\ \Delta_t\ \Gamma_t \to \mathsf{Tm}\ \Delta_t$$
$$\mathsf{var\ tvzero}\ [\ \sigma\ ,_{\mathsf{t}}\ t\ ]\mathsf{t} = t$$
$$\mathsf{var}\ (\mathsf{tvnext}\ tv)\ [\ \sigma\ ,_{\mathsf{t}}\ t\ ]\mathsf{t} = \mathsf{var}\ tv\ [\ \sigma\ ]\mathsf{t}$$

$$\_[\_]\mathsf{f} \;:\; \mathsf{For}\ \Gamma_t \to \mathsf{Subt}\ \Delta_t\ \Gamma_t \to \mathsf{For}\ \Delta_t$$
$$(\mathsf{R}\ t\ u)\ [\ \sigma\ ]\mathsf{f} = \mathsf{R}\ (t\ [\ \sigma\ ]\mathsf{t})\ (u\ [\ \sigma\ ]\mathsf{t})$$
$$(A \Rightarrow B)\ [\ \sigma\ ]\mathsf{f} = (A\ [\ \sigma\ ]\mathsf{f}) \Rightarrow (B\ [\ \sigma\ ]\mathsf{f})$$
$$(\forall\!\!\forall\ A)\ [\ \sigma\ ]\mathsf{f} = \forall\!\!\forall\ (A\ [\ \mathsf{lf}_{\mathsf{t}}\sigma_{\mathsf{t}}\ \sigma\ ]\mathsf{f})$$

✢ Functorality equations

$$
\begin{aligned}
&\mathsf{id_t} \;:\; \mathsf{Subt}\ \Gamma_t\ \Gamma_t \\
&\mathsf{id_t}\ \{\diamond\mathsf{t}\} = \varepsilon_\mathsf{t} \\
&\mathsf{id_t}\ \{\Gamma_t \triangleright \mathsf{t}^0\} = \mathsf{lf_t}\sigma_\mathsf{t}\ (\mathsf{id_t}\ \{\Gamma_t\}) \\
&\_\circ_\mathsf{t}\_ \;:\; \mathsf{Subt}\ \Delta_t\ \Gamma_t \to \mathsf{Subt}\ \Xi_t\ \Delta_t \to \mathsf{Subt}\ \Xi_t\ \Gamma_t \\
&\varepsilon_\mathsf{t} \circ_\mathsf{t} \beta = \varepsilon_\mathsf{t} \\
&(\alpha\ ,_\mathsf{t}\ x) \circ_\mathsf{t} \beta = (\alpha \circ_\mathsf{t} \beta)\ ,_\mathsf{t}\ (x\ [\ \beta\ ]\mathsf{t})
\end{aligned}
$$

× No need for renamings

$$\begin{aligned}
&\text{data Conp} : \text{Cont} \to \textbf{Set} \ \text{where} \\
&\quad \diamond\text{p} : \text{Conp}\ \Gamma_t \\
&\quad _\triangleright\text{p}^0_ : \text{Conp}\ \Gamma_t \to \text{For}\ \Gamma_t \to \text{Conp}\ \Gamma_t
\end{aligned}$$

$$\begin{aligned}
&_[_]\text{c} : \text{Conp}\ \Gamma_t \to \text{Subt}\ \Delta_t\ \Gamma_t \to \text{Conp}\ \Delta_t \\
&\diamond\text{p}\ [\ \sigma_t\ ]\text{c} = \diamond\text{p} \\
&(\Gamma_p \triangleright\text{p}^0\ A)\ [\ \sigma_t\ ]\text{c} = (\Gamma_p\ [\ \sigma_t\ ]\text{c}) \triangleright\text{p}^0\ (A\ [\ \sigma_t\ ]\text{f})
\end{aligned}$$

$$\_\triangleright\mathrm{tp} \;:\; \mathrm{Conp}\ \Gamma_t \to \mathrm{Conp}\ (\Gamma_t \triangleright \mathrm{t}^0)$$
$$\Gamma \triangleright\mathrm{tp} = \Gamma\ [\ \mathrm{wk_t}\sigma_\mathrm{t}\ \mathrm{id_t}\ ]\mathrm{c}$$

$$
\begin{array}{l}
\textsf{data PfVar} : (\Gamma_t : \textsf{Cont}) \to (\Gamma_p : \textsf{Conp}\ \Gamma_t) \to \textsf{For}\ \Gamma_t \to \textbf{Prop} \\
\quad \textsf{pvzero} : \textsf{PfVar}\ \Gamma_t\ (\Gamma_p \triangleright \textsf{p}^0\ A)\ A \\
\quad \textsf{pvnext} : \textsf{PfVar}\ \Gamma_t\ \Gamma_p\ A \to \textsf{PfVar}\ \Gamma_t\ (\Gamma_p \triangleright \textsf{p}^0\ B)\ A \\
\\
\textsf{data Pf} : (\Gamma_t : \textsf{Cont}) \to (\Gamma_p : \textsf{Conp}\ \Gamma_t) \to \textsf{For}\ \Gamma_t \to \textbf{Prop} \\
\quad \textsf{var} : \textsf{PfVar}\ \Gamma_t\ \Gamma_p\ A \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ A \\
\quad \textsf{app} : \textsf{Pf}\ \Gamma_t\ \Gamma_p\ (A \Rightarrow B) \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ A \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ B \\
\quad \textsf{lam} : \textsf{Pf}\ \Gamma_t\ (\Gamma_p \triangleright \textsf{p}^0\ A)\ B \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ (A \Rightarrow B) \\
\quad \textsf{p}\forall\!\!\forall\textsf{e} : \textsf{Pf}\ \Gamma_t\ \Gamma_p\ (\forall\!\!\forall\ A) \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ (A\ [\ \textsf{id}_t\ ,_t\ t\ ]\textsf{f}) \\
\quad \textsf{p}\forall\!\!\forall\textsf{i} : \textsf{Pf}\ (\Gamma_t \triangleright \textsf{t}^0)\ (\Gamma_p \triangleright \textsf{tp})\ A \to \textsf{Pf}\ \Gamma_t\ \Gamma_p\ (\forall\!\!\forall\ A)
\end{array}
$$

$$\_[\_]\mathrm{pv_t} : \{A : \mathrm{For}\ \Delta_t\} \to \mathrm{PfVar}\ \Delta_t\ \Delta_p\ A \to$$
$$(\sigma : \mathrm{Subt}\ \Gamma_t\ \Delta_t) \to \mathrm{PfVar}\ \Gamma_t\ (\Delta_p\ [\ \sigma\ ]\mathrm{c})\ (A\ [\ \sigma\ ]\mathrm{f})$$

$\mathrm{pvzero}\ [\ \sigma\ ]\mathrm{pv_t} = \mathrm{pvzero}$

$\mathrm{pvnext}\ pv\ [\ \sigma\ ]\mathrm{pv_t} = \mathrm{pvnext}\ (pv\ [\ \sigma\ ]\mathrm{pv_t})$

$$\_[\_]\mathrm{p_t} : \{A : \mathrm{For}\ \Delta_t\} \to \mathrm{Pf}\ \Delta_t\ \Delta_p\ A \to$$
$$(\sigma : \mathrm{Subt}\ \Gamma_t\ \Delta_t) \to \mathrm{Pf}\ \Gamma_t\ (\Delta_p\ [\ \sigma\ ]\mathrm{c})\ (A\ [\ \sigma\ ]\mathrm{f})$$

$\mathrm{var}\ pv\ [\ \sigma\ ]\mathrm{p_t} = \mathrm{var}\ (pv\ [\ \sigma\ ]\mathrm{pv_t})$

$\mathrm{app}\ pf\ pf'\ [\ \sigma\ ]\mathrm{p_t} = \mathrm{app}\ (pf\ [\ \sigma\ ]\mathrm{p_t})\ (pf'\ [\ \sigma\ ]\mathrm{p_t})$

$\mathrm{lam}\ pf\ [\ \sigma\ ]\mathrm{p_t} = \mathrm{lam}\ (pf\ [\ \sigma\ ]\mathrm{p_t})$

$$\text{data Ren} : \text{Conp } \Gamma_t \to \text{Conp } \Gamma_t \to \textbf{Set} \text{ where}$$
$$\text{zeroRen} : \text{Ren} \diamond\text{p } \Gamma_p$$
$$\text{leftRen} : \{A : \text{For } \Delta_t\} \to$$
$$\text{PfVar } \Delta_t \; \Delta_p \; A \to \text{Ren } \Delta_p' \; \Delta_p \to \text{Ren } (\Delta_p' \triangleright\text{p}^0 \; A) \; \Delta_p$$

$$\text{data Subp} : \{\Delta_t : \text{Cont}\} \to \text{Conp } \Delta_t \to \text{Conp } \Delta_t \to \textbf{Prop}$$
$$\varepsilon_\text{p} : \text{Subp } \Delta_p \diamond\text{p}$$
$$\_,\text{p}\_ : \{A : \text{For } \Delta_t\} \to (\sigma : \text{Subp } \Delta_p \; \Delta_p')$$
$$\to \text{Pf } \Delta_t \; \Delta_p \; A \to \text{Subp } \Delta_p \; (\Delta_p' \triangleright\text{p}^0 \; A)$$

$$\_[\_]\sigma_\text{p} : \text{Subp } \{\Delta_t\} \; \Delta_p \; \Delta_p' \to (\sigma : \text{Subt } \Gamma_t \; \Delta_t) \to$$
$$\text{Subp } \{\Gamma_t\} \; (\Delta_p \, [\, \sigma \,]\text{c}) \; (\Delta_p' \, [\, \sigma \,]\text{c})$$
$$\varepsilon_\text{p} \, [\, \sigma_t \,]\sigma_\text{p} = \varepsilon_\text{p}$$
$$(\sigma_p \, ,_\text{p} \, pf) \, [\, \sigma_t \,]\sigma_\text{p} = (\sigma_p \, [\, \sigma_t \,]\sigma_\text{p}) \, ,_\text{p} \, (pf \, [\, \sigma_t \,]\text{p}_\text{t})$$

$$
\begin{aligned}
&\_[\_]\mathsf{p}\ :\ \{A\ :\ \mathsf{For}\ \Delta_t\} \to \mathsf{Pf}\ \Delta_t\ \Delta_p\ A\\
&\quad \to (\sigma\ :\ \mathsf{Subp}\ \{\Delta_t\}\ \Delta_p{}'\ \Delta_p) \to \mathsf{Pf}\ \Delta_t\ \Delta_p{}'\ A\\
&\mathsf{var}\ \mathsf{pvzero}\ [\ \sigma\ ,_{\mathsf{p}}\ pf\ ]\mathsf{p} = pf\\
&\mathsf{var}\ (\mathsf{pvnext}\ pv)\ [\ \sigma\ ,_{\mathsf{p}}\ pf\ ]\mathsf{p} = \mathsf{var}\ pv\ [\ \sigma\ ]\mathsf{p}\\
&\mathsf{app}\ pf\ pf\ [\ \sigma\ ]\mathsf{p} = \mathsf{app}\ (pf\ [\ \sigma\ ]\mathsf{p})\ (pf\ [\ \sigma\ ]\mathsf{p})\\
&\mathsf{lam}\ pf\ [\ \sigma\ ]\mathsf{p} = \mathsf{lam}\ (pf\ [\ \mathsf{wk_p}\sigma_{\mathsf{p}}\ \sigma\ ,_{\mathsf{p}}\ \mathsf{var}\ \mathsf{pvzero}\ ]\mathsf{p})\\
&\mathsf{p}\forall\forall\mathsf{e}\ pf\ [\ \sigma\ ]\mathsf{p} = \mathsf{p}\forall\forall\mathsf{e}\ (pf\ [\ \sigma\ ]\mathsf{p})\\
&\mathsf{p}\forall\forall\mathsf{i}\ pf\ [\ \sigma\ ]\mathsf{p} = \mathsf{p}\forall\forall\mathsf{i}\ (pf\ [\ \mathsf{wk_t}\sigma_{\mathsf{p}}\ \sigma\ ]\mathsf{p})
\end{aligned}
$$

$$\mathsf{id_p} \ : \mathsf{Subp}\ \{\Delta_t\}\ \Delta_p\ \Delta_p$$
$$\mathsf{id_p}\ \{\Delta_p = \diamond\mathsf{p}\} = \varepsilon_\mathsf{p}$$
$$\mathsf{id_p}\ \{\Delta_p = \Delta_p \triangleright\mathsf{p}^0\ x\} = \mathsf{lf_p}\sigma_\mathsf{p}\ (\mathsf{id_p}\ \{\Delta_p = \Delta_p\})$$
$$\_\circ_\mathsf{p}\_ \ : \{\Gamma_p\ \Delta_p\ \Xi_p \ : \mathsf{Conp}\ \Delta_t\} \to \mathsf{Subp}\ \{\Delta_t\}\ \Delta_p\ \Xi_p \to \mathsf{Subp}\ \{\Delta_t$$
$$\varepsilon_\mathsf{p} \circ_\mathsf{p} \beta = \varepsilon_\mathsf{p}$$
$$(\alpha\ ,_\mathsf{p}\ pf) \circ_\mathsf{p} \beta = (\alpha \circ_\mathsf{p} \beta)\ ,_\mathsf{p}\ (pf\ [\ \beta\ ]\mathsf{p})$$

```
record Con : Set where
  constructor con
  field
    t : Cont
    p : Conp t
```

```
record Sub (Γ : Con) (Δ : Con) : Set where
  constructor sub
  field
    t : Subt (Con.t Γ) (Con.t Δ)
    p : Subp {Con.t Γ} (Con.p Γ) ((Con.p Δ) [ t ]c)
```

```
pf [ σ ]p = (pf [ Sub.t σ ]pₜ) [ Sub.p σ ]p
```

$$\mathrm{id}\ : \mathrm{Sub}\ \Gamma\ \Gamma$$
$$\mathrm{id}\ \{\Gamma\} = \mathrm{sub}\ \mathrm{id_t}\ (\mathrm{substP}\ (\mathrm{Subp}\ \_)\ (\equiv\mathrm{sym}\ []\mathrm{c\text{-}id})\ \mathrm{id_p})$$
$$\_\circ\_\ : \mathrm{Sub}\ \Delta\ \Xi \to \mathrm{Sub}\ \Gamma\ \Delta \to \mathrm{Sub}\ \Gamma\ \Xi$$
$$\mathrm{sub}\ \alpha_t\ \alpha_p \circ \mathrm{sub}\ \beta_t\ \beta_p =$$
$$\quad \mathrm{sub}\ (\alpha_t \circ_\mathrm{t} \beta_t)\ (\mathrm{substP}\ (\mathrm{Subp}\ \_)\ (\equiv\mathrm{sym}\ []\mathrm{c\text{-}\circ})\ (\alpha_p\ [\ \beta_t\ ]\sigma_\mathrm{p})\ \circ_\mathrm{p}\ \beta_p$$

$$\_\triangleright\mathrm{t}\ : \mathrm{Con} \to \mathrm{Con}$$
$$\Gamma\ \triangleright\mathrm{t} = \mathrm{con}\ ((\mathrm{Con.t}\ \Gamma)\ \triangleright\mathrm{t}^0)\ (\mathrm{Con.p}\ \Gamma\ \triangleright\mathrm{tp})$$
$$\_\triangleright\mathrm{p}\_\ : (\Gamma\ : \mathrm{Con}) \to \mathrm{For}\ (\mathrm{Con.t}\ \Gamma) \to \mathrm{Con}$$
$$\Gamma\ \triangleright\mathrm{p}\ A = \mathrm{con}\ (\mathrm{Con.t}\ \Gamma)\ (\mathrm{Con.p}\ \Gamma\ \triangleright\mathrm{p}^0\ A)$$

id	$\mathsf{Sub}\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)$
id_t	$\mathsf{Subt}\ \Gamma_t\ \Gamma_t$
id_p	$\mathsf{Subp}\ \Gamma_p\ \Gamma_p$
sub	$(\sigma : \mathsf{Subt}\ \Gamma_t\ \Delta_t)$
	$\to \mathsf{Subp}\ \Gamma_p\ (\Delta_p[\sigma]\mathsf{c})$
	$\to \mathsf{Sub}\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)\ (\mathsf{con}\ \Delta_t\ \Delta_p)$

$$\mathsf{id} = \mathsf{sub}\ \mathsf{id}_t\ \mathbf{?} \quad \textit{with} \quad \mathbf{?} : \mathsf{Subp}\ \Gamma_p\ (\Gamma_p[\mathsf{id}_t]\mathsf{c})$$

id	$\mathsf{Sub}\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)$
id_t	$\mathsf{Subt}\ \Gamma_t\ \Gamma_t$
id_p	$\mathsf{Subp}\ \Gamma_p\ \Gamma_p$
sub	$(\sigma : \mathsf{Subt}\ \Gamma_t\ \Delta_t)$ $\to \mathsf{Subp}\ \Gamma_p\ (\Delta_p[\sigma]c)$ $\to \mathsf{Sub}\ (\mathsf{con}\ \Gamma_t\ \Gamma_p)\ (\mathsf{con}\ \Delta_t\ \Delta_p)$

$$\mathsf{id} = \mathsf{sub}\ \mathsf{id}_t\ \mathbf{?} \quad \textit{with} \quad \mathbf{?} : \mathsf{Subp}\ \Gamma_p\ (\Gamma_p[\mathsf{id}_t]c)$$

$$\mathsf{Subp}\ \Gamma_p\ (\Gamma_p[\mathsf{id}_t]c) \not\equiv \mathsf{Subp}\ \Gamma_p\ \Gamma_p$$

id	Sub (con Γ_t Γ_p) (con Γ_t Γ_p)
id_t	Subt Γ_t Γ_t
id_p	Subp Γ_p Γ_p
sub	(σ : Subt Γ_t Δ_t) $\to$ Subp Γ_p ($\Delta_p[\sigma]$c) $\to$ Sub (con Γ_t Γ_p) (con Δ_t Δ_p)

$$\text{id} = \text{sub}\ \text{id}_t\ \textbf{?} \quad \textit{with} \quad \textbf{?} : \text{Subp}\ \Gamma_p\ (\Gamma_p[\text{id}_t]\text{c})$$

$$\text{Subp}\ \Gamma_p\ (\Gamma_p[\text{id}_t]\text{c}) \not\equiv \text{Subp}\ \Gamma_p\ \Gamma_p$$

$$\text{subst} : \{A : \textbf{Set}\}(P : A \to \textbf{Prop})\{a\ a' : A\} \to a \equiv a' \to P\ a \to P\ a'$$